Analysis of decomposed topology optimization problem

B. Radi [1], A. Makrizi [2]

[1] *Faculty of Sciences and Technology Errachidia*
BP : 509, Boutalamine, Errachidia, Morocco.
Bouchaib.radi@yahoo.fr

[2] *GGDA, Faculty of Sciences Ben M'sik*
Boulevard Cdt Driss Harti
BP : 7655, Ben M'sik, Casablanca, Morocco.
a_makrizi@yahoo.fr

RÉSUMÉ. En optimisation topologique, on est souvent amener à résoudre un problème de grande taille ou parfois un problème dont le domaine est hétérogène, ainsi la méthode de décomposition du domaine paraît comme une bonne alternative pour résoudre de tel problème. Dans ce papier, nous présentons une nouvelle approche de l'optimisation topologique en utilisant la méthode décomposition en sous-domaines pour résoudre de tel problème. Ensuite, nous proposons une méthode de résolution du problème obtenu.

ABSTRACT. In topology optimization problems, we are often forced to deal with large-scale numerical problems, so that domain decomposition occurs naturally. Consider a typical topology optimization problem, the minimum compliance problem of a linear isotropic elastic continuum structure, in which the constraints are the partial differential equations of linear elasticity. We subdivide the problem into two sub-problems posed on non-overlapping sub-domains, each of them has boundary data that depends on the solution of the other sub-problem. We determine the unknown data through an optimization problem in which the discrepancy between an appropriate defined functional of the difference between solutions in sub-domains is minimized. This article presents a new formulation of the minimum compliance problem based on the domain decomposition methods, and then we propose an algorithm to solve the founding problem.

MOTS-CLÉS : Optimisation topologique, SIMP, Méthodes de décomposition du domaine, méthode du gradient, éléments finis.

KEYWORDS: Topology optimization, sub-domains method, compliance, elastic structure, gradient method, finite element.

1.Introduction

The topology optimization has for objective to find an optimal shape without any a priori assumption about its topology, i.e., on the nature and the connectivity of elements which constitute it. Mathematically, the topology optimization problem takes the form:

$$\min_{\omega \in \Omega} f(u(\omega), \omega) \qquad [1]$$

s.t :

$$\begin{cases} g_i(u(\omega), \omega) \leq 0 & 1 \leq i \leq m \\ h_j(u(\omega), \omega) = 0 & 1 \leq j \leq n \end{cases}$$

Where u is the solution of the state equation $L(u(\omega), \omega) = 0$, f is the objective function, g_i and h_j are the functions defining the constraints, in practice they are implicit and nonlinear functions in ω.

One can find various methods of topology optimization in the literature for solving the problem [1], methods based on the shape gradient, evolutionary methods [Sigmund, 1997, Chapman, 1994, Kane, 1996], and methods which employ a material distribution approach for a fixed reference domain, especially the homogenization methods [Bendsœ et al., 1988, Allaire et al., 1996]. We are interested to the power-law materials also called SIMP (Solid Isotropic Material with Penalization method) which has seen widespread academic use and has proven very popular and extremely tempting to solve practical applications [Bendsœ, 1995].

In spite of its effectiveness in structural design, topology optimization is not yet largely widespread in the industry, the principal reason is that the topology optimization problem is a large scale optimization problem; it is characterized by a very significant number of design variables, which amplifies the difficulty of its resolution. It is common to introduce 1000 to 10000 design variables to solve a real problem, thus the computation time is typically very high since the problem requires repeated solution of finite element analysis of the equilibrium equations.

However, during two last decades, the parallel computers knew a great evolution, in particular in computing power and storage capacity [Radi et al., 1998]. Domain decomposition methods (called also sub-domains methods) are a valuable approach when solving partial differential equation (PDE) problems on parallel computers [El Hami et al., 1996, Radi et al., 1994]. Any domain decomposition method is based on the assumption that the given computational domain is partitioned into sub-domains which may or may not overlap. Next, the original problem can be reformulated upon each sub-domain, yielding a family of sub-problems of reduced size, that are coupled one to another through the values of the unknown solution at sub-domain interfaces.

Reviewing the literature, it seems that the application of parallel computing in topology optimization is rare, and devoted only to the discrete case [Vemaganti et al., 2004, Mahdavi et al., 2005], there is no mathematical formulation of the topology optimization problem in the continuum case. Thus, The main objective of the present work is to propose a new mathematical formulation of the minimum compliance problem of an isotropic linear elastic structure based on domain decomposition methods when the design domain is partitioned into two non-overlapping sub-domains, the domain decomposition method for the problem of linear elasticity is then based on a constrained minimization problem for which the objective functional measures the jumps in the solution across the interface between sub-domains, the constraints are the partial differential equations.

The remainder of this paper is organized as follows. Section 2 describes the formulation of the topology optimization problem. In section 3, the equivalent formulation is given when the design domain is partitioned into two non-overlapping sub-domains. The finite element approximations is done in section 4. In section 5, the proposed algorithm is presented and some numerical results are given to validate our approach.

2. Description and formulation of the topology optimization problem

The topology optimization problem is a nonlinear optimization problem, often non convex, the objective function depends on a state variable describing the operational mode and the design variables determine the shape and topology, the state variable must satisfy a boundary value problem, here we deals with typical problem of topology optimization which consists in minimizing the compliance of an isotropic linear elastic structure.

Consider an elastic body in the configuration region $\Omega \subset R^d (d = 2,3)$ with boundary Γ. The problem of linear elasticity is given as follows:

$$\begin{array}{ll} Find & u : \Omega \to R^d \, such that \\ \left\{ \begin{array}{ll} (-div\sigma(u))_i = f_i & \Omega, i = 1,...,d \\ u = \varphi_D & \Gamma_D \\ \sum_{j=1}^{d} \sigma_{ij}(u) n_j = (\varphi_N)_i & \Gamma_N, i = 1,...,d. \end{array} \right. \end{array} \qquad [2]$$

where L is a second-order linear elliptic operator, n denotes the unit outward normal vector on Γ. f is the vector of volume forces acting on the body, φ_D is the given displacement on the portion of the domain boundary Γ_D, while φ_N are the tractions applied on the complementary part Γ_N, and $\sigma = (\sigma_{ij})_{1 \le i,j \le d}$ the stress tensor. Take, for simplicity, $\Gamma_N = \varnothing$ and $\varphi_D = 0$, thus the system of equations of linear elasticity [2] becomes the Dirichlet boundary value problem [3] where $\varepsilon = (\varepsilon_{ij})_{1 \le i,j \le d}$ is the strain tensor.

$$\begin{array}{lll} Find & u:\Omega \to R^d & suchthat \\ \begin{cases} (-div\sigma(u))_i = f_i & \Omega & 1 \le i \le d \\ u_i = 0 & \Gamma & 1 \le i \le d \end{cases} & & \end{array} \qquad [3]$$

The variational formulation of [3] reads:

$$\text{Find } u \in H_0^1(\Omega)^d \text{ such that: } a(u,v) = l(v) \quad \forall v \in H_0^1(\Omega)^d \qquad [4]$$

where $a(u,v) = \int_\Omega E_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)d\Omega,$ and $l(v) = \int_\Omega fvd\Omega$.

The Korn's inequality states that there exists a constant $C_\Omega > 0$ such that [Duvaut et al. 1976]:

$$\sum_{i,j=1}^{d} \int_\Omega (\varepsilon_{ij}(v))^2 d\Omega \ge C_\Omega \|v\|^2_{H_0^1(\Omega)^d} \quad \forall v \in H_0^1(\Omega)^d \qquad [5]$$

Hence, the form a is coercive, furthermore it is easily seen that a (resp. l) is bilinear and continuous in $H_0^1(\Omega)^d$ (respectively linear and continuous in $H_0^1(\Omega)^d$, when the problem [4] admits a unique solution $u \in H_0^1(\Omega)^d$ by a straightforward application of the Lax-Milgram theorem. Consequently the minimum compliance (maximum global stiffness) problem takes the following form in the SIMP approach [Bendsœ et al, 2003]:

$$\begin{cases} \min_\rho l(u) \\ a_\rho(u,v) = l(v) \qquad \forall v \in U \subseteq H_0^1(\Omega)^d \\ E_{ijkl}(x) = \rho^p(x)E^0_{ijkl} \end{cases} \qquad [6]$$

with the following constraints on ρ: $\int_\Omega \rho(x)d\Omega \le V$, $0 < \rho_{\min} < \rho(x) \le 1$ for all x in Ω.

U is the set of admissible displacements, V is a limit on the amount of material at our disposal, E^0_{ijkl} represents the material properties of a given isotropic material, ρ which is interpreted as a density of material is the design variable and p is the penalty factor which penalizes intermediate densities in order to end up with (nearly) 'solid and void' distributions. Normally, one writes $E_{ijkl} \in L^\infty(\Omega)$ to indicate the relevant functional space for our problem, unfortunately, in this case, the problem [6] lacks existence of solutions in its general continuum setting. To ensure existence of solutions, the power-law approach must be combined with a perimeter constraint, a gradient constraint or with filtering techniques [Sigmund et al., 1998]. Here we use a gradient constraint by which we mean the norm of the function ρ in the Sobolev space $H^1(\Omega)$, see [Bendsœ et al., 2003]:

$$\|\rho\|_{H^1(\Omega)} = [\int_\Omega (\rho^2 + \|\nabla\rho\|^2)d\Omega] \le M \quad where \quad 1 < p < \frac{d}{d-2} \quad (\Omega \subset R^d)$$

where $\|\nabla\rho\|^2 = \sum_i (\frac{\partial\rho}{\partial x_i})^2$.

Bendsœ has proved existence of solutions when including this bound in the minimum compliance problem [Bendsœ et al., 2003]. Thus, we will choose the new formulation due to Bendsœ:

$$\begin{cases} \min\limits_{u,\rho\in H^1(\Omega)} & l(u) \\ a_\rho(u,v) = l(v) & \forall v \in U \end{cases} \qquad [7]$$

with the last constraints on ρ, and the gradient constraint is bounded, where: $a_\rho(u,v) = \int_\Omega \rho^p(x) E^0_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) d\Omega$.

3. Domain decomposition method for the topology optimization problem

Let Ω be a bounded domain in R^d where d = 2, 3 with Lipschitz boundary Γ. We further suppose that Ω is partitioned into two non-overlapping sub-domains Ω_1 and Ω_2 with interface Γ_0. Let $\Gamma_i = \overline{\Omega}_i \cap \Gamma$, i=1,2. The problem of linear elasticity [4] can be written as:

$$\begin{array}{lll} Find & u_i \in H^1_{\Gamma_i}(\Omega_i)^d, i=1,2 & suchthat \\ & a_{\rho_1}(u_1,v_2) = (f,v_1)_{\Omega_1} + (g,v_1)_{\Gamma_0} & \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d \\ & a_{\rho_2}(u_2,v_2) = (f,v_2)_{\Omega_2} - (g,v_2)_{\Gamma_0} & \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^d \\ & u_1 = u_2 & \Gamma_0 \end{array} \qquad [8]$$

where

$$g_l = \sum_{j=1}^{d} \sigma_{lj}(u_1) n^1_j = -\sum_{j=1}^{d} \sigma_{lj}(u_2) n^2_j \quad and \quad g = (g_l)_{1\le l\le d} \quad with \quad (g,v)_{\Gamma_0} = \int_{\Gamma_0} gv d\Gamma_0$$

The existence and uniqueness of the solution to both problems [8.1] and [8.2] is a straightforward consequence of the Lax-Milgram theorem. For an arbitrary choice for the control g, the solutions u_1 and u_2 of the problem [8.1] and the problem [8.2], respectively, do not agree with the solution u of [4] in the respective sub-domains. The discrepancy is due to the fact that for an arbitrary choice of g, we have that $u_1 \neq u_2$ along Γ_0, even in a weak sense. In addition, there exists clearly a choice of g, namely such that the solutions of the problems [8.1] and [8.2] coincide with the solution of [4] on the corresponding sub-domains. Thus, we consider a functional that measures the jumps of solutions across the interface between sub-domains with a penalty term to regularize the problem, put:

$$J_\delta(u_1,u_2,g) = \frac{1}{2}\int_{\Gamma_0} (u_1-u_2)^2 d\Gamma_0 + \frac{\delta}{2}\int_{\Gamma_0} g^2 d\Gamma_0$$

Let $(\rho_1, \rho_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$ then we consider the following optimization problem:

$$\min_g J_\delta(u_1, u_2, g)$$
$$\begin{cases} a_{\rho_1}(u_1, v_1) = (f, v_1)_\Omega + (g, v_1)_{\Gamma_0} & \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d \\ a_{\rho_2}(u_2, v_2) = (f, v_2)_\Omega - (g, v_2)_{\Gamma_0} & \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^d \end{cases} \quad [9]$$

Let the admissibility set of [9] be defined by:

$$U_{ad} = \left\{ (u_1, u_2, g) \in H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma_2}(\Omega_2)^d \times L^2(\Gamma_0)^d / (9.1) \,\&\, (9.2) satisfied \,\&\, J_\delta(u_1, u_2, g) < \infty \right\} \quad [10]$$

We give the first main theoretical result of this paper: The problem [7] can be equivalently reformulated as:

$$\begin{cases} \min\limits_{u_1, u_2, \rho_1, \rho_2} l_1(u_1) + l_2(u_2) \\ \min\limits_g J_\delta(u_1, u_2, g) \\ a_{\rho_1}(u_1, v_1) = (f, v_1)_{\Omega_1} + (g, v_1)_{\Gamma_0} & \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d \\ a_{\rho_2}(u_2, v_2) = (f, v_2)_{\Omega_2} + (g, v_2)_{\Gamma_0} & \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^d \end{cases} \quad [11]$$

with the following constraints on ρ_i:

$$\sum_{i=1}^{2} \int_{\Omega_i} \rho_i(x) d\Omega_i \le V \quad 0 < \rho_i \le 1 \quad \forall x \in \Omega_i \quad i = 1, 2,$$

and the gradient constraint $\|\rho_i\|_{H^1(\Omega_i)} \le M_i$, i= 1,2, where : $l_i(u_i) = \int_{\Omega_i} f u_i d\Omega_i$. The Lagrangian of the problem [11] is:

$$\begin{aligned} &L(u_1, u_2, \rho_1, \rho_2, g, \lambda_1, \lambda_2, \Lambda_1, \Lambda_2, \lambda_1^+, \lambda_1^-, \lambda_2^-) = \varepsilon_{\sigma,\delta}(u_1, u_2, \rho_1, \rho_2, g) - a_{\rho_1}(u_1, \lambda_1) \\ &+ (f, \lambda_1)_{\Omega_1} + (g, \lambda_1)_{\Gamma_0} - a_{\rho_2}(u_2, \lambda_2) + (f, \lambda_2)_{\Omega_2} - (g, \lambda_2)_{\Gamma_0} + \Lambda_1 \left(\int_{\Omega_1} \rho_1(x) d\Omega_1 - V_1 \right) \\ &+ \Lambda_2 \left(\int_{\Omega_2} \rho_2(x) d\Omega_2 - V_2 \right) + \int_{\Omega_1} \lambda_1^+(x)(\rho_1(x) - 1) d\Omega_1 + \int_{\Omega_1} \lambda_1^-(x)(\rho_{\min}(x) - \rho_1) d\Omega_1 \\ &+ \int_{\Omega_2} \lambda_2^+(x)(\rho_2(x) - 1) d\Omega_2 + \int_{\Omega_2} \lambda_2^-(x)(\rho_{\min}(x) - \rho_1) d\Omega_2 \end{aligned} \quad [12]$$

We apply the necessary conditions for finding stationary points of L, we find:

$$\frac{\partial L}{\partial \lambda_1} = 0 \quad \Rightarrow \quad a_{\rho_1}(u_1, v_1) = (f, v_1)_{\Omega_1} + (g, v_1)_{\Gamma_0} \qquad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^2$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \quad \Rightarrow \quad a_{\rho_2}(u_2, v_2) = (f, v_2)_{\Omega_2} + (g, v_2)_{\Gamma_0} \qquad \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^2$$

$$\frac{\partial L}{\partial u_1} = 0 \quad \Rightarrow \quad a_{\rho_1}(\xi, \lambda_1) = (f, \xi)_{\Omega_1} + \frac{1}{\sigma}(u_1 - u_2, \xi)_{\Gamma_0} \qquad \forall \xi \in H^1_{\Gamma_1}(\Omega_1)^2$$

$$\frac{\partial L}{\partial u_2} = 0 \quad \Rightarrow \quad a_{\rho_2}(\xi, \lambda_2) = (f, \xi)_{\Omega_2} - \frac{1}{\sigma}(u_1 - u_2, \xi)_{\Gamma_0} \qquad \forall \xi \in H^1_{\Gamma_2}(\Omega_2)^2$$

$$\frac{\partial L}{\partial g} = 0 \quad \Rightarrow \quad (g, r)_{\Gamma_0} = -\frac{1}{\sigma}(\lambda_1 - \lambda_2, r)_{\Gamma_0} \qquad \forall r \in L^2(\Gamma_0)^2$$

$$\frac{\partial L}{\partial \rho_i} = 0 \quad \Rightarrow \quad \frac{\partial E_{ijkl}}{\partial \rho_i}\varepsilon_{ij}(u_i)\varepsilon_{kl}(u_i) = \Lambda_i + \lambda_i^+ - \lambda_i^- \qquad i = 1, 2$$

With the switching conditions: $\lambda_i^+ \geq 0$, $\lambda_i^- \geq 0$, $\lambda_i^-(\rho_{\min} - \rho_i(x)) = 0$; $\lambda_i^+(\rho_i(x) - 1) = 0$; i = 1,2. For intermediate densities $(\rho_{\min} < \rho < 1)$ the last condition can be written as

$p\rho(x)^{p-1} E^0_{ijkl}\varepsilon_{ij}(u_i)\varepsilon_{kl}(u_i) = \Lambda_i \quad i = 1, 2.$ Thus, we put an update scheme for the density.

4. Finite element approximations

Let W_1^h, W_2^h denote finite element spaces such that $W_1^h \subset H^1_{\Gamma_1}(\Omega_1)^2$ and $W_2^h \subset H^1_{\Gamma_2}(\Omega_2)^2$. We assume that the usual approximations properties hold, i.e., for i = 1,2, there exists an integer k and a constant C such that:

$$\inf_{v^h \in W_i^h} \left\| v - v^h \right\|_{1,\Omega_i} \leq Ch^m \left\| v \right\|_{m+1,\Omega_i} \qquad \forall v \in H^1_{\Gamma_i}(\Omega_i)^2 \cap H^{m+1}(\Omega_i)^2 \qquad [13]$$

and $0 \leq m \leq k$. The finite element approximations of solutions of the optimality system are defined as follows. Seek $u_1^h \in W_1^h$, $u_2^h \in W_2^h$, $\lambda_1^h \in W_1^h$ and $\lambda_2^h \in W_2^h$ such that

$$\begin{aligned}
a_{\rho_1}(u_1^h, v^h) &= (f, v^h)_{\Omega_1} - \frac{1}{\delta}(\lambda_1^h - \lambda_2^h, v^h)_{\Gamma_0} && \forall v^h \in W_1^h \\
a_{\rho_2}(u_2^h, v^h) &= (f, v^h)_{\Omega_2} + \frac{1}{\delta}(\lambda_1^h - \lambda_2^h, v^h)_{\Gamma_0} && \forall v^h \in W_2^h \qquad [14] \\
a_{\rho_1}(\lambda_1^h, \xi^h) &= (u_1^h - u_2^h, \xi^h)_{\Gamma_0} && \forall \xi^h \in W_1^h \\
a_{\rho 2}(\lambda_2^h, \xi^h) &= -(u_1^h - u_2^h, \xi^h)_{\Gamma_0} && \forall \xi^h \in W_2^h
\end{aligned}$$

We obtain the error estimate for the finite element approximations through the use of the Brezzi-Rappaz-Raviart theory. Let

$$X = H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma_1}(\Omega_1)^2 \times H^1_{\Gamma_2}(\Omega_2)^2 \times H^1_{\Gamma_2}(\Omega_2)^2,$$
$$Y = H^1_{\Gamma_1}(\Omega_1)^{2*} \times H^1_{\Gamma_1}(\Omega_1)^{2*} \times H^1_{\Gamma_2}(\Omega_2)^{2*} \times H^1_{\Gamma_2}(\Omega_2)^{d*},$$
$$Z = L^2(\Gamma_0)^2 \times L^2(\Gamma_0)^2 \times L^2(\Gamma_0)^2 \times L^2(\Gamma_0)^2,$$
$$X^h = W_1^h \times W_1^h \times W_2^h \times W_2^h,$$

Where $H^1_{\Gamma_i}(\Omega_i)^{2*}$ is the dual space of $H^1_{\Gamma_i}(\Omega_i)^2$ for i=1,2. Let Λ be a positive compact interval containing $1/\delta$ and defines the operator $B \in L(Y, X)$ as follows: For $(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) \in Y$ and $(\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2) \in X$, $B(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) = (\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2)$ if and only if

$$a_{\rho_i}(\tilde{u}_i, v) = (\tilde{Q}_i, v)_{\Omega_i} \quad \forall v \in H^1_{\Gamma_i}(\Omega_i)^2, \quad a_{\rho_i}(\tilde{\lambda}_i, r) = (\tilde{P}_i, r)_{\Omega_i} \quad \forall r \in H^1_{\Gamma_i}(\Omega_i)^2, \quad i = 1, 2.$$

Also the discretized operator $B_h \in L(Y, X^h)$ is defined by: $B_h(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) = (\tilde{u}_1^h, \tilde{\lambda}_1^h, \tilde{u}_2^h, \tilde{\lambda}_2^h)$ if and only if

$$a_{\rho_i}(\tilde{u}_i^h, s^h) = (\tilde{Q}_i, s^h)_{\Omega_i} \quad \forall s^h \in W_i^h \quad et \quad a_{\rho_i}(\tilde{\lambda}_i^h, r^h) = (\tilde{P}_i, r^h)_{\Omega_i} \quad \forall r^h \in W_i^h \quad i = 1, 2.$$

We now define the operator G from $\Lambda \times X$ to Y by: for $(\frac{1}{\delta}, (\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2)) \in \Lambda \times X$ and $(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) \in Y$, $G(\frac{1}{\delta}, (u_1, \lambda_1, u_2, \lambda_2)) = (\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2)$ if and only if

$$(\tilde{Q}_1, s)_{\Omega_1} = -(f, s)_{\Omega_1} + \frac{1}{\delta}(\lambda_1 - \lambda_2, s)_{\Gamma_0} \qquad \forall s \in H^1_{\Gamma_1}(\Omega_1)^2,$$
$$(\tilde{P}_1, r)_{\Omega_1} = -(u_1 - u_2, r)_{\Gamma_0} \qquad \forall r \in H^1_{\Gamma_1}(\Omega_1)^2,$$
$$(\tilde{Q}_2, s)_{\Omega_2} = -(f, s)_{\Omega_2} - \frac{1}{\delta}(\lambda_1 - \lambda_2, s)_{\Gamma_0} \qquad \forall s \in H^1_{\Gamma_2}(\Omega_2)^d,$$
$$(\tilde{P}_2, r)_{\Omega_2} = (u_1 - u_2, r)_{\Gamma_0} \qquad \forall r \in H^1_{\Gamma_2}(\Omega_2)^2.$$

If we assume that the regularity property of partial differential equations holds, then we have the following result: Let $(u_1, u_2, \lambda_1, \lambda_2)$ denote the solution of the optimality system and let $(u_1^h, u_2^h, \lambda_1^h, \lambda_2^h)$ denote the solutions of the discrete optimality system. Assume the approximation property [13] holds and $(u_1, \lambda_1) \in H^{m+1}(\Omega_1)^2 \cap H^1_{\Gamma_1}(\Omega_1)^2$ and $(u_2, \lambda_2) \in H^{m+1}(\Omega_2)^2 \cap H^1_{\Gamma_2}(\Omega_2)^2$ for $1 \le m < k$, then

$$\left\|u_1 - u_1^h\right\|_{1,\Omega_1} + \left\|\lambda_1 - \lambda_1^h\right\|_{1,\Omega_1} + \left\|u_2 - u_2^h\right\|_{1,\Omega_2} + \left\|\lambda_2 - \lambda_2^h\right\|_{1,\Omega_2} \le Ch^m \left\|f\right\|_{m-1,\Omega} \quad [15]$$

where C is independent of h.

5. Proposed algorithm and numerical result

We consider the conjugate gradient method. This methode is defined as follows: given a starting guess ρ_1^0, ρ_2^0 and g^0, let

$$g^{(n+1)} = g^{(n)} - \alpha \frac{\partial \varepsilon}{\partial g} \quad for \quad n = 1, 2, ... \qquad [16]$$

where α is a step size, or

$$g^{(n+1)} = (1 - \delta\alpha) g^{(n)} - \alpha(\lambda_1^{(n)} - \lambda_2^{(n)}) \big| \Gamma_0 \qquad [17]$$

In summary, the proposed algorithm is given as follows:

Step1: Choose $\alpha, \rho_1^{(0)}, \rho_2^{(2)}$ and $g^{(0)}$

Step2 : For $n = 0, 1, 2, ...$

1. Determine $u_1^{(n)}$ and $u_2^{(n)}$ from

$$a_{\rho_1}(u_1^{(n)}, v_1) = (f, v_1)_{\Omega_1} + (g^{(n)}, v_1)_{\Gamma_0} \qquad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^2$$
$$a_{\rho_2}(u_2^{(n)}, v_2) = (f, v_2)_{\Omega_2} + (g^{(n)}, v_2)_{\Gamma_0} \qquad \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^2$$

2. Determine $\lambda_1^{(n)}, \lambda_2^{(n)}$ from

$$a_{\rho_1}(\lambda_1^{(n)}, R) = (f, R)_{\Omega_1} + \frac{1}{\sigma}(u_1^{(n)} - u_2^{(n)}, R)_{\Gamma_0} \qquad \forall R \in H^1_{\Gamma_1}(\Omega_1)^2$$
$$a_{\rho_2}(\lambda_2^{(n)}, R) = (f, R)_{\Omega_2} - \frac{1}{\sigma}(u_1^{(n)} - u_2^{(n)}, R)_{\Gamma_0} \qquad \forall R \in H^1_{\Gamma_2}(\Omega_2)^2$$

Step3: Determine $g^{(n+1)}$ from : $g^{(n+1)} = (1 - \delta\alpha) g^{(n)} - \alpha(\lambda_1^{(n)} - \lambda_2^{(n)}) \big| \Gamma_0$

Step4: Convergence test.

The first numerical test of this algorithm is the beam treated by Sigmund [Sigmund, 1997] (see figure 1. The beam is decomposed in two equal beams. Figure 2 gives the result of the direct topology optimization of this structure using the Matlab code proposed in [Sigmund, 1997]. Figures 3 and 4 give the result of the proposed gradient method using our developed Matlab code. The stop test is based on the maximum norm and equal to 10^{-6}. One can show that if the two substructures are assembled, we don't find the result of the figure 2 form the initial structure, because firstly, the uniqueness of the solution of the topology optimization is not shown in any reference and secondly, the subdomain approach adopted in this work

belong to the force approach: the unknown variable at the interface is a forces. So, the finding result stay reasonable and good.

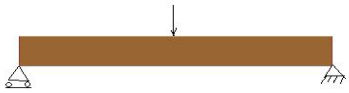

Figure1. The studied structure.

Figure2. The result of the direct topology optimization.

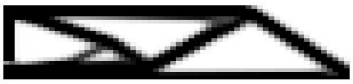

Figure3. The left side of the structure using the our approach.

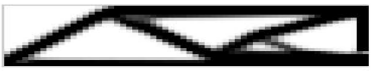

Figure4. The right side of the structure using of our approach.

6. Conclusion

To treat industrial problems with great dimension using the finite element approximation per example, we propose in this article the subdomain methods based on the decomposition of the conception domain in many subdomains with reasonable dimension and we show that this approach is useful in topology optimization. Some theoretical justifications are given and the equivalence between the original problem and the decomposed one is showed. The obtained formulation in the present work is a problem of optimization of which the resolution involves special techniques of optimization algorithm. The gradient method is proposed to solve the decomposed problem. This method is very adequate to parallel computer and had proven a very good performance in parallel computing.

The approach suggested, in this paper, can be extended to other problems of topology optimization using fictitious materials (SIMP model). The first numerical results encourage us to apply the proposed approach to complex structures.

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